ISRAEL JOURNAL OF MATHEMATICS **127** (2002), 1–17

DIMENSION SPECTRA OF SELF-AFFINE SETS

BY

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ABSTRACT

The dimension spectrum $H(\delta)$ is a function characterizing the distribution of dimension of sections. Using the multifractal formula for sofic measures, we show that the dimension spectra of irreducible self-affine sets (McMullen's Carpet) coincide with the modified Legendre transform of the free energy $\Psi_{\mathrm{d}}(\beta)$. This variational relation leads to the formula of Hausdorff dimension of self-affine sets, $\max(\delta + H(\delta)) = \Psi_{\mathrm{d}}(\eta)$, where η is the logarithmic ratio of the contraction rates of the affine maps.

1. Introduction

A self-affine set is composed of affine-contracted parts of itself. McMullen [8] calculated the Hausdorff dimension of self-affine sets, or McMullen's Carpets. Kenyon and Peres [6] calculated the Hausdorff dimension of more general self-affine sets (partially self-affine sets), which correspond to sofic shifts while McMullen's Carpets correspond to full shifts. They obtained their result by approximating the partially self-affine set by McMullen's Carpet.

In this paper, we investigate the dimension spectrum of self-affine sets and establish its variational formula, as well as the relation to its Hausdorff dimension. The dimension spectrum $H(\delta)$ of a set S is the Hausdorff dimension of the set of the heights where the horizontal section of the set S has box dimension δ. In [9, 10], we have shown that the dimension spectrum $H(\delta)$ of the limit set of linear cellular automata coincides with the Legendre transform of the free energy for

Received January 20, 2000 and in revised form February 28, 2000

dimension spectrum $\Psi_{\mathrm{d}}(\beta)$. Its proof uses a special property of cellular automata, and does not apply to more general fractal sets.

The orthogonal projection of the Hausdorff measure on the limit set of linear cellular automata to the vertical or the horizontal axis is a sofic measure ([4]). In [11], we have shown the multifractal formalism for irreducible sofic measures, with singularity α between its minimum $\alpha_{\min}$ and α_0, the value where the singularity spectrum $f(\alpha)$ attains its maximum: $\alpha_{\min} < \alpha \leq \alpha_0$, $\alpha_{\min} = \inf\{\alpha\colon f(\alpha) > 0\}$ and $\alpha_0 = \inf\{\alpha\colon f(\alpha) = \max_{\alpha'} f(\alpha')\}$. There exists a correspondence between singularity α of the sofic measure and the box dimension δ of the horizontal section of the limit set. We apply the multifractal formula for sofic measures to the dimension spectrum $H(\delta)$ and show that the dimension spectrum $H(\delta)$ is given as a modified Legendre transformation of the free energy $\Psi_{\mathrm{d}}(\beta)$, if δ is in the upper region ($\delta_0 \leq \delta < \delta_{\max}$):

$$H(\delta) = \max_{\beta}\left(\Psi_{\mathrm{d}}(\beta) - \frac{\beta\delta}{\eta}\right) \quad (\delta_0 \leq \delta < \delta_{\max}) \tag{1}$$

where the order η is the ratio of logarithms of the vertical contraction rate to that of the horizontal one, $\delta_0 = \sup\{\delta\colon H(\delta) = \max_{\delta'} H(\delta')\}$, and $\delta_{\max} = \sup\{\delta\colon H(\delta) > 0\}$. Using this variational formula, we give the Hausdorff dimension of a self-affine set X, $\dim_H X$, as the summation of the dimension of the horizontal direction δ and the vertical one $H(\delta)$:

$$\dim_H X = \max_{\delta}(\delta + H(\delta)), \tag{2}$$

which is also expressed by the free energy $\Psi(\eta)$.

2. Definition and main results

We consider following self-affine set in R^2 that consists of N-patterns. We divide the unit square into rectangles of a columns and b rows,

$$\left[\frac{p}{a}, \frac{p+1}{a}\right] \times \left[\frac{q}{b}, \frac{q+1}{b}\right] \quad (p = 0, \ldots, a-1,\ q = 0, \ldots, b-1).$$

We denote a-cylinders on the x-axis $\left[\frac{x_1\cdots x_n}{a^n}, \frac{x_1\cdots x_n+1}{a^n}\right]$ and b-cylinders on the y-axis $\left[\frac{y_1\cdots y_n}{b^n}, \frac{y_1\cdots y_n+1}{b^n}\right]$ by $[x_1 \ldots x_n]$ and $[y_1 \ldots y_n]$, respectively. Let f_{pq} be the orientation preserving affine map from the unit square to the rectangle $[p] \times [q]$. Let g_k ($k = 1, \ldots, N$) be a map from $(p, q) \in \{0, \ldots, a-1\} \times \{0, \ldots, b-1\}$ to $g_k(p, q) \in \{0, \ldots, N\}$. Let $\{X_1, \ldots, X_N\}$ be a family of non-empty compact sets which satisfies the following set of equations:

$$
\begin{aligned}
X_0 &= \emptyset, \\
X_1 &= \bigcup_{p,q} f_{pq}(X_{g_1(p,q)}), \\
&\vdots \\
X_N &= \bigcup_{p,q} f_{pq}(X_{g_N(p,q)}).
\end{aligned} \tag{3}
$$

Then we refer, by **self-affine sets**, to one of the sets in $\{X_1, \ldots, X_N\}$. The number $\eta = \log b / \log a$ is the **order** of the self-affine set.

From now on, we deal with the self-affine set $X = X_1$ and assume that $a \geq b$ and that $\{X_1, \ldots, X_N\}$ is **irreducible**, i.e., for each pair of patterns X_i and X_j, X_i contains an affine contracted pattern of X_j.

A **level set** L_y of X is defined by

$$L_y = \{x\colon (x,y) \in X\}. \tag{4}$$

The dimension spectrum $H(\delta)$ of X is the Hausdorff dimension of the set of y's where the box dimension of level set L_y equals δ:

$$H(\delta) = \dim_H \{y\colon \dim_b L_y = \delta\}. \tag{5}$$

We define the free energy for dimension spectrum $\Psi_{\mathrm{d}}(\beta)$ by

$$\Psi_{\mathrm{d}}(\beta) = \lim_{n \to \infty} \frac{\log \sum_{y_1 \ldots y_n} N(y_1 \ldots y_n)^\beta}{\log b^n}, \tag{6}$$

where $N(y_1 \ldots y_n)$ is the number of affine contracted patterns of $X_1, \ldots, X_N$ contained in X with vertical side $[y_1 \ldots y_n]$:

$$N(y_1 \ldots y_n) = \sharp \left\{ q\colon \left(\frac{y_1 \ldots y_n}{b^n}, \frac{y_1 \ldots y_n + 1}{b^n} \right) \times \left(\frac{q}{a^n}, \frac{q+1}{a^n} \right) \cap X = \emptyset \right\}. \tag{7}$$

From the dimension spectrum $H(\delta)$, the Hausdorff dimension of a self-affine set is represented as follows.

THEOREM 2.1: *Let $\eta = \log b / \log a$ be the order of a self-affine set X. If the self-affine set is irreducible, its Hausdorff dimension is given by*

$$\dim_H X = \max_\delta (\delta + H(\delta)) = \Psi_{\mathrm{d}}(\eta). \tag{8}$$

Theorem 2.1 indicates that the dimension of a self-affine set is given as the summation of those in the horizontal direction, δ, and in the vertical direction, $H(\delta)$.

Note: Replacing the box dimension $\dim_b L_y$ in (5) with the Hausdorff dimension $\dim_H L_y$ does not change the result stated in Theorem 2.1: if $\delta_0 \leq \delta < \delta_{\max}$, then

(9) $$\dim_H\{y\colon \dim_b L_y\} = \dim_H\{y\colon \dim_H L_y\}.$$

Equation (9) follows from the following facts. We assume $\delta_0 \leq \delta < \delta_{\max}$.

(i) Replacing the box dimension in (5) with the lower box dimension $\underline{\dim_b} L_y$ does not change its value ([11]):

(10) $$\dim_H\{y\colon \dim_b L_y = \delta\} = \dim_H\{y\colon \underline{\dim_b} L_y = \delta\}.$$

(ii) A well-known inequality:

(11) $$\dim_H L_y \leq \underline{\dim_b} L_y.$$

(iii) There exist a measure μ^β, defined by (34), and a set E, defined by Definition 3.7, such that

(12) $$\mu^\beta\left(E \cap \{y\colon \dim_b L_y = \delta\}\right) = 1$$

and, for any y in E,

(13) $$\dim_H L_y = \underline{\dim_b} L_y.$$

3. Multifractal formula of Sofic measures and variational formula of dimension spectra

In this section, we relate the dimension spectrum $H(\delta)$ to the singularity spectrum $f(\alpha)$ of a sofic measure on the vertical axis. The **singularity spectrum** $f(\alpha)$ of a measure μ is the Hausdorff dimension of the set of points where the **singularity**

$$\lim_{n\to\infty} \frac{\log \mu([y_1 \cdots y_n])}{\log b^{-n}}$$

equals α:

(14) $$f(\alpha) = \dim_H \left\{y\colon \lim_{n\to\infty} \frac{\log \mu([y_1 \cdots y_n])}{\log b^{-n}} = \alpha\right\}.$$

In many cases, especially for quasi-multiplicative measures, the singularity spectrum $f(\alpha)$ coincides with the Legendre transform of the free energy $\Psi(\beta)$ ([2]). The **free energy of a measure** μ is defined by

$$\Psi(\beta) = \lim_{n\to\infty} \frac{\log \sum_{y_1\ldots y_n} (\mu[y_1\ldots y_n])^\beta}{\log b^{-n}}. \tag{15}$$

The equality of the singularity spectrum and the Legendre transform of the free energy

$$f(\alpha) = \inf_\beta (\Psi(\beta) - \alpha\beta) \tag{16}$$

is called the multifractal formula. The multifractal formula holds for quasi-multiplicative measures ([2]).

The sofic measure as well as the semi-group measure are natural measures on sofic systems. The sofic measure of a sofic system with b-symbols is defined as follows ([4]). Let $A_0, \ldots, A_{b-1}$ be non-negative square matrices of the same size. Let v_0 be a non-negative row vector. Let u be a non-negative right eigenvector of $(A_0 + \cdots + A_{b-1})$ and λ be its eigenvalue. The sofic measure of a cylinder $[y_1 \ldots y_n]$ is given by

$$\mu([y_1\ldots y_n]) = \frac{v_0 A_{y_1}\ldots A_{y_n} u}{\lambda^n v_0 u}. \tag{17}$$

The sofic measure is **irreducible** if the summation of the matrices $(A_0 + \cdots + A_{b-1})$ is irreducible.

For irreducible sofic measures, the multifractal formula holds at the left half of the graph of the singularity spectrum:

$$f(\alpha) = \inf_{\beta\geq 0} (\Psi(\beta) - \alpha\beta) \quad (\alpha_{\min} < \alpha \leq \alpha_0), \tag{18}$$

where

$$\alpha_{\min} = \inf\{\alpha\colon f(\alpha) > 0\} \tag{19}$$

and

$$\alpha_0 = \sup\{\alpha\colon f(\alpha) = \max_{\alpha'} f(\alpha')\}. \tag{20}$$

The multifractal formula does not hold at the right side of the graph of $f(\alpha)$ for sofic measures in general ([11]).

The semi-group measure on a r-symbol sofic system is defined as follows ([7]). Let f_k ($k = 1,\dots,r$) be maps from $\{1,\dots,N\}$ to $\{1,\dots,N\}$. Let A be the non-negative N by N square matrix whose components are given by

$$A_{ij} = \sharp\{k\colon f_k(i) = j\}. \tag{21}$$

Let u be a non-negative right eigenvector of A and λ be its eigenvalue. We assume that u_1 is positive. The semi-group measure μ of a cylinder $[z_1 \dots z_n]$ is given by

$$\mu\left([z_1 \dots z_n]\right) = \frac{u_{f_{z_n}\cdots f_{z_1}(1)}}{\lambda^n u_1}. \tag{22}$$

If the matrix A is irreducible, the multifractal formula holds for the whole range of α ([3]).

To relate the dimension spectrum $H(\delta)$ and its free energy $\Psi_{\mathrm{d}}(\beta)$ to $f(\alpha)$ and $\Psi(\beta)$, we introduce a semi-group measure on X as follows, whose projection onto the y-axis gives a sofic measure.

Let the $N \times N$ **transition matrix** A be defined by

$$A_{ij} = \sharp\{(p,q)\colon g_i(p,q) = j\} \quad (i,j = 1,\dots,N), \tag{23}$$

where g_i: $\{0,\dots,a-1\} \times \{0,\dots,b-1\} \to \{0,\dots,N\}$ is defined in the beginning of section 2 and indicates that the pattern at $[p]\times[q]$ in X_i is the affine contracted pattern of $X_{g_i(p,q)}$ with $X_0 = \emptyset$.

The element of the transition matrix A_{ij} represents the number of affine-contracted pattern X_j's in pattern X_i. Irreducibility of $\{X_1,\dots,X_N\}$ implies that of the transition matrix A.

Let u be a non-negative right eigenvector of the transition matrix A with respect to its Frobenius eigenvalue λ. The semi-group measure M on X is given by

$$M\left([x_1 \dots x_n] \times [y_1 \dots y_n]\right) = \frac{u_j}{\lambda^n u_1}, \tag{24}$$

if $[x_1 \dots x_n] \times [y_1 \dots y_n] \cap X$ is the affine-contracted pattern of X_j.

Let μ be the orthogonal projection of the above semi-group measure onto the y-axis. The measure μ of a b-adic-cylinder $[y_1 \dots y_n]$ is represented by the partial transition matrices, $A_0,\dots,A_{b-1}$, as

$$\mu\left([y_1 \cdots y_n]\right) = \frac{v_0 A_{y_1} \cdots A_{y_n} u}{\lambda^n}, \tag{25}$$

where $v_0 = (1,0,\dots,0)$, and the **partial transition matrices** $A_0,\dots,A_{b-1}$ are defined by

$$(A_k)_{ij} = \sharp\{p\colon g_i(p,k) = j\}. \tag{26}$$

A partial transition matrix A_k indicates the number of patterns in the k-th stage $[\frac{k}{b}, \frac{k+1}{b}]$.

Using partial transition matrices, the number of rectangles $[x_1 \dots x_n] \times [y_1 \dots y_n]$ of X_1 containing a non-empty pattern with vertical side $[y_1 \dots y_n]$, $N(y_1 \dots y_n)$, is represented as

$$N(y_1 \dots y_n) = v_0 A_{y_1} \dots A_{y_n} \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}, \tag{27}$$

where $\begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}$ is the vertical vector with all components unity. From (25) and (27), the singularity α of the measure μ at a point $y = 0.y_1 y_2 \dots$ in the b-adic expansion, is given by

$$\alpha = \lim_{n \to \infty} \frac{\log \frac{N(y_1 \cdots y_n)}{\lambda^n}}{\log b^n}. \tag{28}$$

Let $\eta = \log b / \log a$ be the order of a self-affine set X. Noting that the box dimension δ of a level set L_y is given by

$$\delta = \dim_b L_y = \lim_{n \to \infty} \frac{\log N(y_1 \dots y_n)}{\log a^n}, \tag{29}$$

we have, from (28),

$$\delta + \eta\alpha = \log \lambda / \log a. \tag{30}$$

Concerning the free energies, (15), (28), (29) and (30) lead to the relation between the free energy of singularity spectrum $\Psi(\beta)$ and the free energy of dimension spectrum $\Psi_{\mathrm{d}}(\beta)$:

$$\Psi(\beta) + \Psi_{\mathrm{d}}(\beta) = \beta \frac{\log \lambda}{\log \beta}. \tag{31}$$

From (30) and (31), multifractal formalism (18) applies to the dimension spectrum as follows.

Lemma 3.1: *Let $\delta_{\max} = \sup\{\delta\colon H(\delta) > 0\}$ and $\delta_0 = \sup\{\delta\colon H(\delta) = 1\}$. The dimension spectrum $H(\delta)$ with $\delta_0 \le \delta < \delta_{\max}$ is given as a modified Legendre transform of the free energy $\Psi_{\mathrm{d}}(\beta)$:*

$$H(\delta) = \inf_{\beta \ge 0} \left(\Psi_{\mathrm{d}}(\beta) - \frac{\beta\delta}{\eta} \right). \tag{32}$$

Before proving Theorem 2.1, we illustrate the situation by an example.

Example 3.2: We consider a self-affine set with $N = 2$ (2 patterns), $a = 3$ (horizontal contraction rate is $\frac{1}{3}$), $b = 2$ (vertical contraction rate is $\frac{1}{2}$) and $g_1(0,1) = g_1(2,1) = g_2(1,1) = 0$, $g_1(0,0) = g_1(1,1) = g_1(2,0) = g_2(1,0) = 1$, $g_1(1,0) = g_2(0,0) = g_2(0,1) = g_2(2,0) = g_2(2,1) = 2$, i.e., X_1 contains the affine contracted X_1 at $\left[\frac{0}{3},\frac{1}{3}\right] \times \left[\frac{0}{2},\frac{1}{2}\right]$, $\left[\frac{1}{3},\frac{2}{3}\right] \times \left[\frac{1}{2},\frac{2}{2}\right]$, $\left[\frac{2}{3},\frac{3}{3}\right] \times \left[\frac{0}{2},\frac{1}{2}\right]$, the affine-contracted X_2 at $\left[\frac{1}{3},\frac{2}{3}\right] \times \left[\frac{0}{2},\frac{1}{2}\right]$, while X_2 contains the affine-contracted X_1 at $\left[\frac{1}{3},\frac{2}{3}\right] \times \left[\frac{0}{2},\frac{1}{2}\right]$, and the affine-contracted X_2 at $\left[\frac{0}{3},\frac{1}{3}\right] \times \left[\frac{0}{2},\frac{1}{2}\right]$, $\left[\frac{2}{3},\frac{3}{3}\right] \times \left[\frac{0}{2},\frac{1}{2}\right]$, $\left[\frac{2}{3},\frac{3}{3}\right] \times \left[\frac{1}{2},\frac{2}{2}\right]$. The self-affine sets X_1 and X_2 are shown in Figure 1. X_1 is composed of three affine-transformed X_1's and one X_2 while X_2 is composed of one X_1 and four X_2's. The order of the self-affine set is given by $\eta = \log b / \log a = \log 2 / \log 3$.

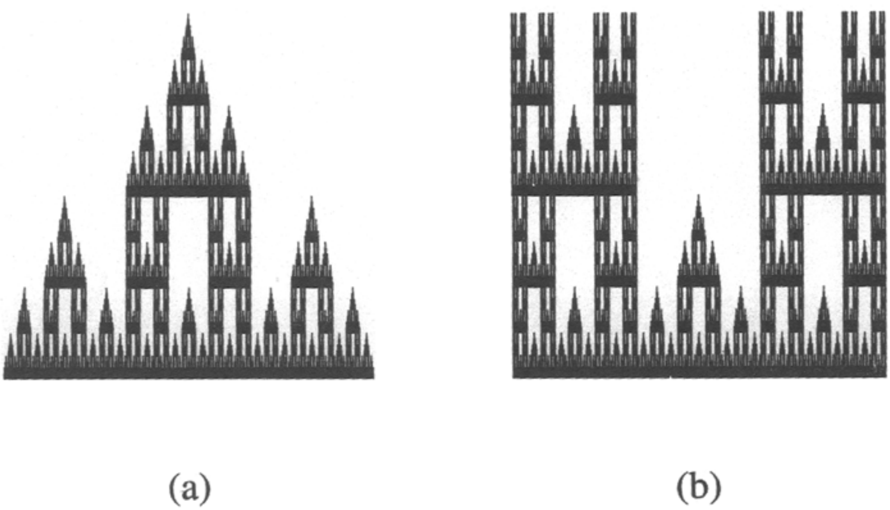
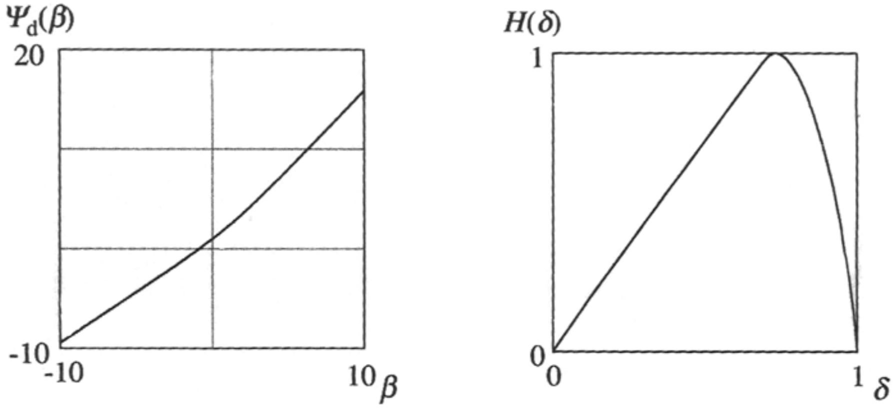


Figure 1. Self-affine sets X_1 (triangle pattern (a)) and X_2 (square pattern (b)). X_1 consists of three contracted X_1's and one contracted X_2, while X_2 consists of one X_1 and four X_2's. The contraction rate is $\frac{1}{3}$ in the horizontal direction, and $\frac{1}{2}$ in the vertical direction. The order η is given by $\eta = \frac{\log \frac{1}{2}}{\log \frac{1}{3}} = \frac{\log 2}{\log 3}$. In the lower half, X_1 contains two contracted X_1 and one contracted X_2, while X_2 consists of one X_1 and two X_2's, which gives the partial transition matrix $A_0 = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$. Similarly, we obtain $A_1 = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$.

Counting affine transformed patterns in the lower part and the upper part separately, we obtain the partial transition matrices $A_0 = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$ and $A_1 = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$.

From the partial transition matrices, we calculate the dimension spectrum of X_1 through the Legendre transformation of the free energy $\Psi_{\mathrm{d}}(\beta)$. In Figure 2, numerical calculation of the free energy $\Psi_{\mathrm{d}}(\beta)$ of X_1 and its Legendre transformation are shown. As stated in Theorem 2.1, the right side of the graph of the Legendre transformation of $\Psi_{\mathrm{d}}(\beta)$ coincides with the dimension spectrum $H(\delta)$ of X_1. However, the left side of the graph deviates from $H(\delta)$: while the Legendre transform of $\Psi_{\mathrm{d}}(\beta)$ is positive between between 0 and $\log 2/\log 3$, no level set L_y has dimension in that interval. The Legendre transformation of $\Psi_{\mathrm{d}}(\beta)$ of X_2 is shown in Figure 3 by a bold solid line. It touches the x-axis at $\log 2/\log 3$.



(a) (b)

Figure 2. (a) The free energy $\Psi_{\mathrm{d}}(\beta)$ of the self-affine set X_1.

(b) The Legendre transform of $\Psi_{\mathrm{d}}(\beta)$. It deviates from $H(\delta)$, since no level set has dimension between 0 and $\log 2/\log 3$.

The Hausdorff dimension of X_1 and X_2 is given by

$$\Psi_{\mathrm{d}}(\eta) = \Psi_d\left(\frac{\log 2}{\log 3}\right) = 1.7509\ldots.$$

To calculate the Hausdorff dimension of self-affine set X, we extend Billingsley's lemma ([1]) to the higher dimensional situation. A pair of a family of cylinder sets $\{c_n(p)\}$ and a probability measure ν is said to be regular if it satisfies the following four conditions.

(r-1) In two of $\{c_n(p)\}$, either one contains the other, or their intersection does not decrease their diameters:
Let $A = \{c_m(q)\colon c_n(p) \cap c_m(q) \neq c_n(p) \text{ and } c_n(p) \cap c_m(q) \neq c_m(q)\}$; then $\left| c_n(p) \bigcap \bigcup_{c_m(q) \in A} c_m(q) \right| = |c_n(p)|$.

(r-2) There exist a natural number L and a positive real number D such that any set S with diameter less than D can be covered by L cylinders of diameter less than that of S.

(r-3) For any point p, there exists a decreasing series of cylinders $c_n(p)$ with bounded ratio of diameters converging to the point $\{p\}$:
$\forall p$, $\exists c_n(p)$, $p \in c_n(p)$, $c_n(p) \supset c_{n+1}(p)$, $\lim_{n\to\infty} |c_n(p)| = 0$ and

$$\liminf_{n\to\infty} \frac{|c_{n+1}(p)|}{|c_n(p)|} > 0.$$

(r-4) If two cylinder sets $c_n(p)$ and $c_m(q)$ do not contain each other, their intersection is the null set of ν:

$$c_n(p) \not\subset c_m(q) \quad \text{and} \quad c_m(q) \not\subset cn(p) \Rightarrow \nu(c_n(p) \cap c_m(q)) = 0.$$

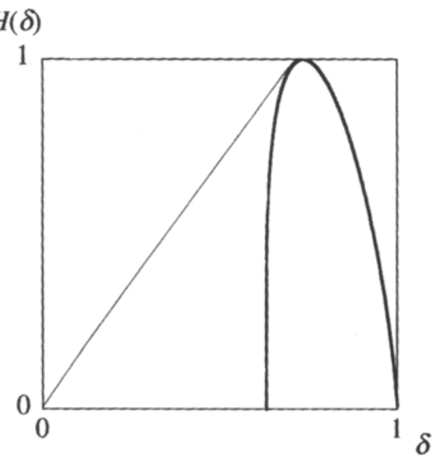


Figure 3. The Legendre transform of X_2's free energy (bold line) and that of X_1's (thin line). They coincide at their right-side slope.

LEMMA 3.3: *Suppose a family of cylinders $\{c_n(p)\}$ and a probability measure ν satisfies the above conditions (r-1) to (r-4). Then the following estimates of the dimension of a set S hold:*

(a) $\dim_H S \leq \sup_{p\in S} \liminf_{n\to\infty} \frac{\log \nu(c_n(p))}{\log|c_n(p)|}$;

(b) *if* $\nu(S) > 0$, *then* $\dim_H S \geq \inf_{p\in S} \liminf_{n\to\infty} \frac{\log \nu(c_n(p))}{\log|c_n(p)|}$.

Proof of Lemma 3.3: To prove (a), we show that $\mathcal{H}^{\alpha+\varepsilon_1}(S) < \infty$ for any $\varepsilon_1 > 0$, with

$$\alpha = \sup_{p\in S} \liminf_{n\to\infty} \frac{\log \nu(c_n(p))}{\log |c_n(p)|}.$$

From the definition of α, for any $\varepsilon_2 > 0$, there exists a ε_2-covering of S, $\{V_i\}$, such that $|V_i|^{\alpha+\varepsilon_1} \leq \nu(V_i)$. From the property (r-4), we may assume that the intersection of V_i and V_j $(i \neq j)$ is the null set of ν. Hence

$$\mathcal{H}^{\alpha+\varepsilon_1}_{\varepsilon_2} \leq \sum_i |V_i|^{\alpha+\varepsilon_1} \leq \sum_i \nu(V_i) \leq \nu(S) \leq 1. \tag{33}$$

(b) is proved in a similar manner. ∎

We construct cylinder sets satisfying the conditions (r-1) to (r-3). Let $z = (x, y)$ be a point in the self-affine set X. We define cylinder sets $c_n(z)$ by

$$c_n(z) = [x_1 \ldots x_m] \times [y_1 \ldots y_n],$$

where m is the integer part of

$$\eta n = \frac{\log b}{\log a} n.$$

Obviously the family of cylinder sets defined above, $\{c_n(z)\}$, satisfies the above regularity conditions (r-1) to (r-3).

We construct the Gibbs measure M^β on X satisfying (r-4), and apply Lemma 3.3 to it. First, we define the Gibbs measure μ^β of μ defined in (25) onto the y-axis by

$$\mu^\beta([y_1 \ldots y_n]) = \lim_{n\to\infty} \frac{\sum_{z_1\ldots z_m} \mu([y_1\ldots y_n z_1 \ldots z_m])^\beta}{\sum_{w_1\ldots w_n, z_1\ldots z_m} \mu([w_1\ldots w_n z_1\ldots z_m])^\beta}. \tag{34}$$

The measure M^β is given by

(35) $M^\beta(c_n(z)) =$

$$\lim_{k\to\infty} \sum_{p_1\ldots p_k} \frac{\mu(y_1 \ldots y_n p_1 \ldots p_k)^\beta v_i A_{y_{m+1}} \ldots A_{y_n} A_{p_1} \ldots A_{p_k} \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}}{\sum_{w_1\ldots w_{n+k}} \mu(w_1 \ldots w_n w_{n+1} \ldots w_{n+k})^\beta N(y_1 \ldots y_n p_1 \ldots p_k)},$$

where the region $[x_1 \dots x_m] \times [y_1 \dots y_m]$ contains the pattern X_i, and v_i is the row vector with i-th component 1 and the other components 0. Note that μ^β is the orthogonal projection of M^β to the y-axis.

Before calculating the Hausdorff dimension of X, we need a variation of Billingsley's lemma, which is proved in the same way as in Lemma 3.3.

LEMMA 3.4: *Let $\{S_q\}$ be a division of a set S: $S = \bigcup_q S_q$ and $S_i \cap S_j = \emptyset$. Suppose cylinder sets $\{c_n(p)\}$ and probability measure ν satisfy regularity conditions (r-1) to (r-4). We assume that the n-th stage cylinders $\{c_n(p)\}$ have the same diameter. Then we have an estimate of the dimension of S:*

$$\dim_H S \leq \sup_p \liminf_{n\to\infty} \frac{\log \frac{\sum_{c_n(p): p\in S_q} \nu(c_n(p))}{\sharp\{c_n(p):\, p\in S_q\}}}{\log |c_n(p)|}.$$

The proof of Theorem 2.1 consists of three lemmas.

LEMMA 3.5: *If the self-affine set X is irreducible, we have*

$$\dim_H X \geq \max_\delta (\delta + H(\delta)).$$

Before proving Lemma 3.5, we define the t-regular set, which is introduced in [11]. We note that there exists a product of partial transition matrices that has a similar property to positive matrices, if their summation is irreducible.

PROPOSITION 3.6 ([11]): *We assume that the summation of the partial transition matrices, $(A_0 + \cdots + A_{b-1})$, is irreducible. There exists a family of subsets of $\{1,\dots,N\}$, $g_1,\dots,g_L$, with $g_1 \cup \cdots \cup g_L = \{1,\dots,N\}$, which satisfies the following two conditions.*

(1) For any partial transition matrix A_k and any group g_I, there exists a group g_J such that, for any i in g_I, if $(A_k)_{ij}$ is positive, then j belongs to g_J, that is, A_k maps the group g_I into the group g_J.

(2) There exists a collection of products of partial transition matrices $\{B_1,\dots,B_M\}$ such that, for each i, there is a B_s with $(B_s)_{ii} > 0$ and such that, for a group g_I containing i, the positive entries of B_s in $g_I \times g_I$ forms a rectangular shape, i.e., $(B_s)_{i_1 j_1} > 0$ and $(B_s)_{i_2 j_2} > 0$ with $i_1, i_2, j_1, j_2 \in g_I$ implies $(B_s)_{i_1 j_2} > 0$ and $(B_s)_{i_2 j_1} > 0$.

Without loss of generality, regarding a certain length of symbol sequences as one symbol if necessary, we may assume that $B_1,\dots,B_M$ in Proposition 3.6 are contained in the partial transition matrices , $A_1,\dots,A_b$, and we denote them as $A_{t_1},\dots,A_{t_M}$. We define the shift σ_{y_1} on the set of groups $\{g_1,\dots,g_l\}$ as follows

$\sigma_{y_1}(g_I)$ is the group such that $i \in g_I$ and $(A_{y_1})_{ij} > 0$ implies $j \in \sigma_{y_1}(g_I)$, i.e., A_{y_1} maps the group g_I into the group $\sigma_{y_1}(g_I)$. Let $t(g_I)$ be the set of t_s's where A_{t_s} has a non-empty set of non-zero elements with rectangular shape in $g_I \times g_I$.

The t-regular set is the set where t_s appears sufficiently often in its expansion and is defined as follows.

Definition 3.7. (t-Regular Set [11]): Let E_k be the set of y's such that the r-adic expansion of y, $y = 0.y_1y_2\ldots$, contains at least one $y_i \in t(\sigma_{y_1\ldots y_{i-1}}(g_1))$ in the first k digits, and at least one $y_i \in t(\sigma_{y_1\ldots y_{i-1}}(g_1))$ in the following $k+1$ digits, and so on:

$$(36)\qquad E_k = \left\{y\colon \forall j \geq 0, \exists i \in \left\{\frac{j(2k+j-1)}{2}+1,\ldots,\frac{(j+1)(2k+j)}{2}\right\}, y_i \in t(\sigma_{y_1\ldots y_{i-1}}(g_1))\right\}.$$

The t-regular set E is the union of E_k's:

$$(37)\qquad E = \bigcup_{k=1}^{\infty} E_k.$$

The t-regular set has the following properties.

LEMMA 3.8 ([11]): *The t-regular set E has full measure of μ^β: $\mu^\beta(E) = 1$.*

Let S_α be the set of points where the singularity of μ equals α. Using the t-regular set and ergodicity of the shift, we have the following lemma.

LEMMA 3.9 ([11]): *If the singularity α is between α_0 and $\alpha_{\min}$, there exists a non-negative β such that $\mu^\beta(S_\alpha) = 1$.*

On the set

$$S_\alpha = \left\{y\colon \frac{\log \mu([y_1\cdots y_n])}{\log b^{-n}} \to \alpha\right\},$$

the singularity spectrum of μ^β is given by $\alpha\beta - \Psi(\beta)$.

LEMMA 3.10: *Let y belong to the set of S_α. Then the singularity of the Gibbs measure μ^β at y for non-negative β is given by*

$$\lim_{n\to\infty} \frac{\log \mu^\beta([y_1\cdots y_n])}{\log |[y_1\cdots y_n]|} = \alpha\beta - \Psi(\beta).$$

Proof of Lemma 3.5: Let E be the t-regular set defined in Definition 3.7. We define the set F by $([1,0]\times E)\cap X$.

Let T_α be the inverse image of S_α in X by orthogonal projection to the y-axis: $T_\alpha = ([1,0]\times S_\alpha)\cap X$, where S_α is the set of points with singularity α. From

Lemmas 3.8 and 3.9, if we take β satisfying $\mu^\beta(S_\alpha)=1$, then $M^\beta(T_\alpha\cap F)=1$, since μ^β is the projection of M^β, and $\mu^\beta(E)=1$.

Let $(x,y)\in T_\alpha\cap F$, $r=\max\{k\colon y_k\in t(\sigma y_1\dots y_{k-1}(g_1)), k<m\}$ and $s=\min\{k\colon y_k\in t(\sigma y_1\dots y_{k-1}(g_1)), k>m\}$. From

$$\frac{\max_i\left(v_0A_{y_1\dots y_r}\right)_i}{\min\{\left(v_0A_{y_1\dots y_r}\right)_i\colon \left(v_0A_{y_1\dots y_r}\right)_i\neq 0\}}\leq a$$

and $\sum_{ij}\left(A_{y_{r+1}\dots y_s}\right)_{ij}\geq 1$ with $i\in\sigma_{y_1\dots y_r}(g_1)$ and $j\in\sigma_{y_1\dots y_s}(g_1)$, and $\left(A_{y_k}\right)_{ij}\leq a$, we have

$$\frac{1}{a^{s-r+2}}v_iA_{y_{m+1}}\dots A_{y_s}\leq\frac{v_0A_{y_1}\dots A_{y_s}}{|v_0A_{y_1}\dots A_{y_m}|},\tag{38}$$

where $i\in\sigma_{y_1\dots y_m}(g_1)$ and v_i is the row vector with i-th component 1 and the other components 0. Since y is in E, r is smaller than

$$n<\frac{m+1}{\eta}=\frac{\log a}{\log b}(m+1)$$

for sufficiently large n. Therefore (38) implies

$$\frac{1}{a^{s-r+2}}N(y_1\dots y_m)\leq\frac{N(y_1\dots y_np_1\dots p_k)}{v_iA_{y_{m+1}}\dots A_{y_n}A_{p_1}\dots A_{p_k}\begin{pmatrix}1\\ \vdots\\ 1\end{pmatrix}},\tag{39}$$

and hence, from (35),

$$M^\beta(c_n(z))\leq a^{s-r+2}\frac{\mu^\beta\left([y_1\dots y_n]\right)}{N(y_1\dots y_m)}.\tag{40}$$

Using (40), the singularity of M^β is evaluated as

$$\lim_{n\to\infty}\frac{\log M^\beta(c_n(z))}{\log|c_n(z)|}\geq\lim_{n\to\infty}\frac{\log\mu^\beta([y_1\dots y_n])}{\log b^{-n}}+\lim_{m\to\infty}\frac{\log N(y_1\dots y_m)}{\log a^m}.\tag{41}$$

Letting $\delta=\frac{\log\lambda}{\log a}-\eta\alpha$, from Lemma 3.10, the first term of (41) tends to

$$\begin{aligned}\lim_{n\to\infty}\frac{\log\mu^\beta([y_1\dots y_n])}{\log b^{-n}}&=\beta\alpha-\Psi(\beta)\\&=f(\alpha)\\&=H(\delta),\end{aligned}\tag{42}$$

while the second term of (41) tends to

$$\lim_{m\to\infty}\frac{\log N(y_1\dots y_m)}{\log a^m}=\delta. \tag{43}$$

Since $M^{\beta}(T_\alpha\cap F)=1$, using Billingsley's lemma (Lemma 3.3), the Hausdorff dimension of the self-affine set X is evaluated from below by

$$\dim_H X\ge\dim_H(T_\alpha\cap F)\ge\delta+H(\delta)\quad\left(\delta=\frac{\log\lambda}{\log a}-\eta\alpha\right), \tag{44}$$

or

$$\dim_H X\ge\max_{\delta_0\le\delta<\delta_{\max}}(\delta+H(\delta)). \quad ∎ \tag{45}$$

LEMMA 3.11: $\max_\delta(\delta+H(\delta))=\Psi_{\mathrm{d}}(\eta)$.

Proof Lemma 3.11: Since $(\log\sum_y N(y_1\dots y_n)^{\beta})/\log b^n$ in the right hand side of (6) is a convex function of β, so is $\Psi_{\mathrm{d}}(\beta)$. As the inverse transformation of the modified Legendre transformation in Lemma 3.1, we have

$$\Psi_{\mathrm{d}}(\beta)=\max_\delta\left(H(\delta)+\frac{\delta\beta}{\eta}\right). \tag{46}$$

Setting $\beta=\eta$, we obtain

$$\max_\delta(\delta+H(\delta))=\Psi_{\mathrm{d}}(\eta). \quad ∎ \tag{47}$$

LEMMA 3.12: $\dim_H X\le\Psi_{\mathrm{d}}(\eta)$.

Proof Lemma 3.12: We consider the singularity of M^η. Let L_y be the subset of X where the y-coordinate is $L_y=\{(x',y')\in X\colon y'=y\}$. Since the projection of M^η is μ^η, we have

$$\frac{\dfrac{\log\sum_{c_n(z):\,z\in L_y}M^\eta(c_n(z))}{\log\sharp\{c_n(z)\colon z\in L_y\}}}{\log b^{-n}}=\frac{\log\mu^\eta([y_1\dots y_n])}{\log b^{-n}}+\frac{\log N(y_1\dots y_m)}{\log a^m}. \tag{48}$$

Noting that the ratio of $\mu([y_1\dots y_n])$ to $N(y_1\dots y_n)/a^n$, as well as $\mu^\beta([y_1\dots y_n])$ to $\sum_{w_1\dots w_k}\mu([y_1\dots y_n w_1\dots w_k])^\beta/\sum_{w_1\dots w_k}\mu([w_1\dots w_k])^\beta$ is bounded, the inferior limit of the first term of (48) tends to

$$\liminf_{n\to\infty} \frac{\log \mu^\eta([y_1\dots y_n])}{\log b^{-n}} \tag{49}$$

$$= \liminf_{n\to\infty} \Big[\frac{\log \mu([y_1\dots y_n])^\eta}{\log b^{-n}} - \frac{\log \sum_{z_1\dots z_n} \mu([z_1\dots z_n])^\eta}{\log b^{-n}}\Big]$$

$$= \liminf_{n\to\infty} \Big[-\frac{\log N(y_1\dots y_n)^\eta}{\log b^n} + \frac{\log a^{n\eta}}{\log b^n} + \frac{\log \sum_{z_1\dots z_n} N(z_1\dots z_n)^\eta}{\log b^n} - \frac{\log a^{n\eta}}{\log b^n}\Big]$$

$$= \liminf_{n\to\infty} \Big(\frac{1}{n}\frac{\log N(y_1\dots y_n)}{\log a}\Big) + \Psi_{\mathrm{d}}(\eta).$$

From McMullen's lemma in [8], the inferior limit of the second term of (48) and the first term in (49) together are evaluated as

$$\liminf_{n\to\infty} \Big[-\frac{1}{n}\frac{\log N(y_1\dots y_n)}{\log a} + \frac{\log N(y_1\dots y_m)}{\log a^m}\Big] = \liminf_{n\to\infty} -\frac{\log \frac{N(y_1\dots y_n)^{1/n}}{N(y_1\dots y_m)^{1/m}}}{\log a} \le 0. \tag{50}$$

Therefore, the inferior limit of (48) is evaluated as

$$\liminf_{n\to\infty} \frac{\frac{\log \sum_{c_n(z):\, z\in L_y} M^\eta(c_n(z))}{\log \sharp\{c_n(z):\, z\in L_y\}}}{\log b^{-n}} \le \Psi_{\mathrm{d}}(\eta). \tag{51}$$

By applying a variation of Billingsley's lemma (Lemma 3.4), we have

$$\dim_H X \le \Psi_{\mathrm{d}}(\eta). \quad \blacksquare \tag{52}$$

Theorem 2.1 immediately follows from Lemmas 3.5–3.12.

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